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Numerical Solution of a Bubble Cavitation Problem

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A model of a cavitating bubble, consisting of a second-order ordinary differential equation, is studied. The bubble consists of an adiabatic gas surrounded by a viscous incompressible liquid with surface tension at the liquid-gas interface. Two numerical solution methods are given, one using a regularizing transformation and the other using suitably powerful general-purpose integrators. The numerical results show that the bubble tends to return to its maximum radius on successive oscillations. This indicates that viscosity is not sufficient to account for rapid damping of bubble oscillations.

1. INTRODUCTION

Wentzell [10] has developed the following equation, Eq. (1), to describe the dynamics of a cavitating bubble. The bubble consists of an adiabatic gas surrounded by a viscous, incompressible liquid with surface tension at the liquid–gas interface. The bubble has been hit by a tension wave at $s = 0$. Further details can be obtained in Ref. [10].

$$x \frac{d^2x}{ds^2} + \frac{3}{2}\left(\frac{dx}{ds}\right)^2 + \frac{a}{x}\left(\frac{dx}{ds}\right) + 1 - cx^{-3\gamma} + Dx^{-1} = \left(\frac{P_s}{P_c}\right) e^{-s/s_*}, \tag{1}$$

where

$$x = R(s)/R_0, \qquad s = \left(\frac{\tau}{R_0}\right)\left(\frac{P_c}{\rho}\right)^{1/2}, \qquad a = \frac{4\mu}{R_0(\rho P_c)^{1/2}},$$

$$s_* = \frac{2.9 \times 10^{-5}}{R_0}\left(\frac{P_c}{\rho}\right)^{1/2}, \qquad D = \frac{2\sigma}{R_0 P_c}, \qquad c = 1 + D;$$

at $s = 0$, $x = 1$, $dx/ds = 0$; $R(s)$ is the bubble radius for $s \geqslant 0$ with $R(0) = R_0$. The variable τ is time, in seconds. For the model studied in Ref. [10], $\rho = 1\text{g/cm}^3$,

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$P_c = 10^6$ dyn/cm^2 (1 atm), $\sigma = 72.8$ dyn/cm, $\gamma = 1.4$, $P_s = 5$ atm, $\mu = 10^{-2}$ dyn sec/cm^2.

There is a critical value of R_0, R^*_0, such that for $R_0 > R^*_0$ the bubble will cavitate, but for $R_0 < R^*_0$ the bubble undergoes rather mild oscillations about its initial radius. Numerical experiments have shown that $R^*_0 \approx 1.71 \times 10^{-5}$.

To study the motion of the bubble for $s \geqslant 0$, Eq. (1) is converted to a system of first-order ordinary differential equations (ODE's),

$$\frac{dy_1}{ds} = y_2, \tag{2a}$$

$$\frac{dy_2}{ds} = \left\{\left(\frac{P_s}{P_c}\right) e^{-s/s*} - 1 - \frac{3}{2} y_2^2\right\} \frac{1}{y_1} - \frac{ay_2 + D}{y_1^2} + \frac{c}{y_1^{3\gamma+1}}, \tag{2b}$$

where $y_1(0) = 1$, $y_2(0) = 0$, $P_s/P_c = 5$, $s_* = 0.029/R_0$, $a = 4 \times 10^{-5}/R_0$, $D = 1.456 \times 10^{-4}/R_0$, $c = 1 + D$, and $\gamma = 1.4$.

Wentzell [10] found that for $R_0 = 10^{-3}$, y_1 can become small ($\sim 6 \times 10^{-5}$) while simultaneously $|y_2| = |dy_1/ds|$ becomes large ($\sim 10^8$) for $s \approx 175$. Although y_2 passes continuously from -10^8 to 10^8 for $y_1 \approx 6 \times 10^{-5}$, this occurs within 10^{-8} s-units. Consequently, it would be natural to refer to such points as cusp-like points. In Ref. 10 it was found that successive bubble-radius maxima monotonically decreased. These results were obtained by numerically integrating Eq. (2) through cusp-like points. Wentzell [10] had considerable difficulty in integrating through each cusp-like point, and consequently his numerical results may be suspect after the first cusp-like point, for the following reason. In Eq. (2) the terms Dy_1^{-2}, $-\frac{3}{2} y_2^2 y_1^{-1}$, $-ay_2 y_1^{-2}$, and $cy_1^{-3\gamma-1}$ are not sufficiently large for a long enough s-interval to account for the amount of damping that was observed. Near a cusp-like point these terms are large but this occurs within an interval of approximately 10^{-8} s-units.

As is known, Henrici [5], the accuracy obtained in numerically integrating Eq. (2) depends on (among other things) the magnitudes, and sometimes on the relative magnitudes, of the eigenvalues of the Jacobian matrix of the right-hand side of Eq. (2). At the cusp-like point near $s = 175$, $y_1 \approx 6 \times 10^{-5}$ and $|y_2| \approx 10^8$ so that the two eigenvalues are complex, with real and imaginary parts approximately $\pm 10^{13}$, and hence the system is *not* stiff. Near a cusp-like point very small integration steps are required to maintain integration accuracy.

In this paper, we describe two main approaches to the numerical solution of our bubble cavitation problem. These are (a) integration of the original equations, Eq. (2), by suitably powerful integrators, and (b) modification of the original equations by regularizing transformations designed to smooth out the trajectory near cusp-like points, followed by integration of the regularized equations. In Section 2, two forms of regularizing transformations are discussed, while in Section 3 the numerical results are presented. Our experience suggests that efforts to find and implement suitable regularizations for such nonsmooth problems may not always be warranted, but that a better first approach would be to attempt a solution with one of the powerful general-purpose integrators which have become available in recent years. The numerical results also show that viscosity is not sufficient to account for rapid damping of bubble oscillations, contrary to what is reported by Wentzell [10].

2. The Regularizing Transformations

A technique is required to smooth out the trajectory in the neighborhood of cusp-like points. One such procedure is the method of regularization, commonly used in planetary trajectory computations, when there is a close-encounter with the gravitational source (Baumgarte and Stiefel [2]). These close-encounters are cusp-like points. In the regularizing transformation, the old independent variable, s, in Eq. (2) is to change slowly near a cusp-like point.

Let t be the new independent variable, such that

$$\frac{ds}{dt} = y_1^{\alpha}, \tag{3}$$

where $\alpha > 0$ is a parameter to be determined. The initial value of t is arbitrary and may be taken as zero. Observe that as a cusp-like point is approached s changes slowly compared to t, because y_1 is becoming small. Using Eq. (3), the equations for the bubble cavitation problem, Eq. (2), become

$$\frac{dy_1}{dt} = y_2 y_1^{\alpha}, \tag{4a}$$

$$\frac{dy_2}{dt} = \left\{\left(\frac{P_s}{P_c}\right) e^{-s/s*} - 1 - \frac{3}{2} y_2^2\right\} y_1^{\alpha-1} - \{ay_2 + D\}\, y_1^{\alpha-2} + c y_1^{\alpha-3\gamma-1}. \tag{4b}$$

Two different sets of regularized equations are obtained from Eq. (4). In the first set, called Regularization 1, a specific value is assigned to α, while in the second set, Regularization 2, Eq. (4) is transformed again by way of a change in the dependent variable y_2.

Regularization 1. In Eq. (4b), y_1 is to appear only to a nonnegative power. Consequently, set $\alpha = \max\{3\gamma + 1, 2\} = 3\gamma + 1$, as $\gamma = 1.4$. The regularized equations are

$$\frac{dy_1}{dt} = y_2 y_1^{3\gamma+1}, \tag{5a}$$

$$\frac{dy_2}{dt} = \left\{\left(\frac{P_s}{P_c}\right) e^{-s/s*} - 1 - \frac{3}{2} y_2^2\right\} y_1^{3\gamma} - \{ay_2 + D\}\, y_1^{3\gamma-1} + c, \tag{5b}$$

$$\frac{ds}{dt} = y_1^{3\gamma+1} \tag{5c}$$

with y_1, y_2, and s known at $t = 0$.

Regularization 2. This set of regularized equations is obtained from Eq. (4) by setting $y_3 = y_2 y_1^{\alpha}$. Equation (4) becomes

$$\frac{dy_1}{dt} = y_3, \tag{6a}$$

$$\frac{dy_3}{dt} = \left\{\left[\left(\frac{P_s}{P_c}\right) e^{-s/s*} - 1\right] y_1 - D\right\} y_1^{2\alpha-2} + \left(\alpha - \frac{3}{2}\right) y_3^2 y_1^{-1} - a y_3 y_1^{\alpha-2} + c y_1^{2\alpha-3\gamma-1}, \tag{6b}$$

$$\frac{ds}{dt} = y_1^{\alpha}, \tag{6c}$$

with y_1, y_3, and s known at $t = 0$.

In this form of the regularization, $|y_3|$ will be very much smaller than $|y_2|$ near cusp-like points. If $\alpha = (3\gamma + 1)/2$ then only nonnegative powers of y_1 will occur in Eq. (6b), except for the $y_3^2 y_1^{-1}$ term. However, by setting $\alpha = \frac{3}{2}$ the term $(\alpha - \frac{3}{2}) y_3^2 y_1^{-1}$ is removed from Eq. (6b). Equation (6) can also be derived from Eq. (1), using Eq. (3), and defining $x = y_1$ and $dx/dt = y_3$. We used Regularization 2 in our numerical studies with both of the above values of α.

3. Numerical Results

The following numerical experiments are grouped according to the integrators used. Various computational results are summarized in Tables I and II.

3.1. *Experiments with DIFSUB* [4]

The ODE integrator DIFSUB is due to Gear [4]. His program was modified to be compatible with the WATFIV Compiler (although used with the FORTRAN-H Compiler) and converted to double precision. The calculations were done in double precision on the IBM 360/75 Computer at the University of Waterloo. DIFSUB controls the magnitude of the single-step errors, measured relative to the quantities y_1 and $\max(|z|, 1)$ for y_1 and z, respectively. (Here, z can be y_2, y_3, or s.) An error bound of 10^{-4} was selected. Also, the DIFSUB option for nonstiff systems was used.

DIFSUB was first tried on the problem in its unregularized form (2), with $R_0 = 10^{-3}$. With a minimum allowable step-size of 10^{-12}, the integration failed near the first cusp-like point, stopping with an error message that "corrector convergence could not be achieved." However, when the minimum allowable step-size was reset to 10^{-16}, DIFSUB was able to integrate through the first cusp-like point. These results are given in Table I, along with those originally obtained by Wentzell [10]. Location of each extremum was obtained by solving $y_2 = 0$, using Newton's method, after the extremum had been bracketed by successive integration steps.

TABLE I

Computational Results for a Bubble of Radius $R_0 = 10^{-3}$

		DIFSUB, Ref. [4]					
Bubble extremum	Wentzell, (unregularized equations)	Unregularized equations	Regularization 1 (Eq. (5))	Regularization 2 (Eq. (6) with $\alpha = \frac{3}{2}$)	Regularization 2 (Eq. (6) with $\alpha = (3\gamma + 1)/2$)	GEAR, Ref. [6] (unregularized equations)	EPISODE, Refs. [3, 7, 8] (unregularized equations)
Maximum 1	$y_1 = 73.7$ $s = 102.1$	$y_1 = 73.53$ $s = 105.29$	$y_1 = 73.53$ $s = 105.29$	$y_1 = 73.53$ $s = 105.29$	$y_1 = 73.53$ $s = 105.29$	$y_1 = 73.51$ $s = 105.27$	$y_1 = 73.53$ $s = 105.28$
Minimum 1	$y_1 = 6.24 \times 10^{-5}$ $s = 175.4$	$y_1 = 5.31 \times 10^{-5}$ $s = 174.92$	$y_1 = 5.36 \times 10^{-5}$ $s = 174.94$	$y_1 = 5.34 \times 10^{-5}$ $s = 174.94$	$y_1 = 5.33 \times 10^{-5}$ $s = 174.93$	$y_1 = 5.24 \times 10^{-5}$ $s = 174.84$	$y_1 = 5.24 \times 10^{-5}$ $s = 174.90$
Maximum 2	$y_1 = 44.6$ $s = 212.4$	$y_1 = 72.69$ $s = 241.41$	$y_1 = 72.56$ $s = 241.33$	$y_1 = 72.68$ $s = 241.44$	$y_1 = 72.70$ $s = 241.45$	$y_1 = 73.32$ $s = 241.92$	$y_1 = 72.98$ $s = 241.67$
Minimum 2	$y_1 = 2.83 \times 10^{-3}$ $s = 257.0$	$y_1 = 5.32 \times 10^{-5}$ $s = 307.85$	$y_1 = 5.38 \times 10^{-5}$ $s = 307.66$	$y_1 = 5.34 \times 10^{-5}$ $s = 307.87$	$y_1 = 5.33 \times 10^{-5}$ $s = 307.91$	$y_1 = 5.13 \times 10^{-5}$ $s = 308.92$	$y_1 = 5.04 \times 10^{-5}$ $s = 308.36$

TABLE II

Integration Statistics for Different Integrators and Equations; Bubble of Radius $R_0 = 10^{-3}$

Bubble extremum	DIFSUB, Ref. [4]								GEAR, Ref. [6] (unregularized equations)		EPISODE, Refs. [3, 7, 8] (unregularized equations)	
	Unregularized equations		Regularization 1 (Eq. (5))		Regularization 2 (Eq. (6) with $\alpha = \frac{3}{2}$)		Regularization 2 (Eq. (6) with $\alpha = (3\gamma + 1)/2$)					
	IS[a]	DE[b]	IS	DE	IS	DE	IS	DE	IS	DE	IS	DE
Maximum 1	105	227	254	544	95	216	185	426	95	121	102	112
Minimum 1	645	1660	1035	2376	165	417	386	861	367	752	408	706
Maximum 2	1054	2880	1727	3577	259	659	570	1263	796	1184	848	1260
Minimum 2	1538	4116	2510	5412	333	870	766	1695	1058	1834	1000	1768

[a] IS = integration steps.
[b] DE = derivative evaluations.

The computer results obtained with DIFSUB showed that the bubble bounces, in that nearly the same bubble-radius maxima are attained. This is different from the monotonic decay observed in Ref. 10. Examination of the differential equations (2) shows that as $s \to +\infty$ the forcing function $(P_s/P_c)\, e^{-s/s_*}$ tends to zero. Consequently, the system approaches an unforced system in which $P_s/P_c = 0$. Application of the test of Dulac [1] to the unforced system shows that there are no periodic solutions (limit cycles). The system eventually decays to its equilibrium state ($y_1 = 1, y_2 = 0$), but s would be quite large before the decay would be noticable.

As a further check on the bouncing effect observed using DIFSUB, the differential equations for a bubble of radius $R_0 = 10^{-2}$ were integrated over the interval $0 \leqslant s \leqslant 220$. Within this interval there were 14 bubble-radius maxima and minima. The bubble-radius maxima had values in the range $8.239 \leqslant (y_1)_{\max} \leqslant 8.3456$, while the minima were in the range $1.074 \times 10^{-2} \leqslant (y_1)_{\min} \leqslant 1.149 \times 10^{-2}$.

It should be noted that the computational results of Ref. 10 were obtained with an integrator that did not use single-step error control. As one can see, the results given in column 1 of Table I show that the trajectory rapidly decays after the first minimum is passed. The rapid decay is attributed to numerical integration errors in the vicinity of the first cusp-like point. This demonstrates the danger in integrating ODE's without some control on the single-step errors.

Next, DIFSUB was used on the regularized equations: Regularization 1, Eq. (5), and Regularization 2, Eq. (6). Again, $R_0 = 10^{-3}$ was used. For Eq. (6), both $\alpha = \frac{3}{2}$ and $\alpha = (3\gamma + 1)/2$ were used. These results are given in Table I.

It is of interest to compare the efficiencies of integrating the various equations. In Table II, we give the number of integration steps (IS) and the number of derivative evaluations (DE) to attain each of the bubble extrema for the $R_0 = 10^{-3}$ bubble. Each of these quantities is measured from the beginning of the integration at $s = 0$. The results, using DIFSUB, show that Regularization 2 (Eq. (6) with $\alpha = \frac{3}{2}$) is best while Regularization 1, Eq. (5), is less efficient than the original equations.

Another regularization was also tried with DIFSUB. This is one in which arc-length is used as the independent variable. It has the advantages of being quite general, and of avoiding step-lengths below the roundoff level in the independent variable. However, the reduction in cost was less than 15 %, while the best of the other regularizations reduced the cost by a factor of about 4.7.

3.2. *Experiments with GEAR* [6] *and EPISODE* [3, 7, 8]

As an additional experiment, the unregularized problem (2) was attempted with two other ODE integrators which have been developed more recently than Gear's DIFSUB [4]. The first of these is a package called GEAR [6], which evolved from DIFSUB through a long series of modifications. The second integrator is a newer package called EPISODE [3, 7, 8]. Both of these packages have various method options (depending on the presence or absence of stiffness, among other things). The one chosen here is the standard nonstiff option (MF = 10), as this problem is not a stiff one. The primary difference between the two packages is that EPISODE is based on variable-step multistep formulas, while GEAR (like DIFSUB) is based on fixed-step formulas, with step-size changes accomplished by use of interpolation. Both of these packages vary the order as well as the step-size in a dynamic manner, and include complicated strategies and safeguards for the selection of these on the basis of accuracy and efficiency considerations.

The calculations described below were done on a C.D.C. 7600 computer at the Lawrence Livermore Laboratory. Single precision was used, as this gives nearly the same accuracy as double precision on the IBM 360/75. The single-step errors were measured as those for DIFSUB: the same error bound of 10^{-4} was used.

Both GEAR and EPISODE were able to integrate the problem (2), with $R_0 = 10^{-3}$, to the second minimum with no difficulty. The computational results are given in Table I, while the corresponding computational statistics are given in Table II. EPISODE did a slightly more efficient job than GEAR, as would be expected on the basis of the methods. Some caution was necessary in assuring that the integrators were free to use a sufficiently wide range of step-sizes, Δs. Steps of 10^{-16} to 10^{-15} were necessary near the cusps, but the step-lengths near the maxima were about 10. (The changes in step-length from step to step were not great, however.) Note that this temporarily results in the odd fact that $s + \Delta s = s$ in the computer, but this does not hamper the integrators because $y_i + \Delta y_i \neq y_i$ and the integration of the y_i continues normally.

The locations and values of the maxima and minima obtained with GEAR and EPISODE agree well with those obtained using DIFSUB, in Table I, considering the error controls imposed and the fact that two different computers were used. Each maximum was obtained by Newton's method, as in the DIFSUB runs. (The minima were located accurately enough by inspection of the step data.)

These calculations were repeated with other values of R_0 as well, and in all cases both GEAR and EPISODE were successful on the problem. As a particular case, we mention that of $R_0 = 1.8 \times 10^{-5}$. The first cusp-like point is near $s = 9.576 \times 10^3$, with $y_1 = 1.33 \times 10^{-8}$ and $|y_2| \approx 10^{15}$. Step-sizes near 10^{-27} were required to pass through the cusp.

4. Concluding Remarks

The computational results obtained here indicate that viscosity is not sufficient to account for rapid damping of bubble oscillations. The major effect on rapid damping of bubble oscillations is probably the compressibility of the surrounding fluid, as reported by Keller and Kolodner [9]. It is believed that the effect of viscosity will be important in the initiation of cavitation. Further work on this aspect is under way.

As to the numerical methods used to obtain these results, it would seem that, for this problem at least, the use of an appropriately powerful ODE integrator is a more efficient overall approach than the design of a regularization procedure. Both are successful, but the former requires little more than programming the right-hand sides of the ODE system, while the latter is bound to require a considerably greater human effort. The regularization approach has the potential advantage of requiring less computer time, as Regularization 2 showed. However, for a problem of this size it appears that the savings would be at most a few seconds of computer time. Of course, if one were solving many equations repetitively, such as in solving two-point boundary-value problems by shooting methods, then the design and use of an appropriate regularization would be worthwhile.

As a result of this experience, we make the following general recommendation: when faced with the task of solving (numerically) a problem of a given type, the computational physicist should first attempt a solution with whatever general-purpose solvers are available for the given problem type, and only if that fails pursue more specialized approaches designed around the troublesome features in the problem. This order of procedure has the potential benefit of saving a great deal of human effort. Of course, if the general approach succeeds only at the expense of large computational costs, then a special approach may be called for, and one must make trade off considerations between human time and computer time.

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